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Convex subspace closure of the point shadow of an apartment of a spherical building

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Abstract. Let Σ be the point-line truncation of an augmented J-Grassmann geometry of a spherical building $\mathcal{B}$. We show that if $|J| = 1$, or if $|J| \geq 2$ and some additional conditions hold, then the convex subspace closure in Σ of the point shadow of an apartment of $\mathcal{B}$ is the entire space Σ.

1 Introduction

Let $\mathcal{B}$ be a spherical building over a type set I. Let $J \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the J-Grassmann geometry of $\mathcal{B}$. The geometry Σ can be regarded as a point-line space (see 12.15 of [11] or Lemma 2.4 of [7]). Let $S \subseteq \mathcal{P}$. The *span* or the *subspace closure* of S in Σ is the intersection of all subspaces of Σ containing S. The *convex subspace closure* of the set S in Σ is the intersection of all convex subspaces of Σ containing S.

When $|J| = 1$, the question whether or not the point shadow of an apartment of $\mathcal{B}$ spans the space Σ has been investigated in [4] and [2]. The answer varies depending on the Dynkin diagram of $\mathcal{B}$ and the choice of the set J.

The main result of the present paper is Theorem 1.1 of Section 1.3. This theorem shows that, if the J-Grassmann geometry Σ has rank at least two, and $|J| = 1$, or $|J| \geq 2$ and some additional condition holds, then the convex subspace closure of the point shadow of an apartment of $\mathcal{B}$ in Σ is the entire space Σ. We chose to state Theorem 1.1 in terms of point-line truncations of augmented J-Grassmann geometries (defined in Section 1.3), instead of J-Grassmann geometries, so that we can avoid making exceptions for low-rank cases. If the rank of the J-Grassmann geometry is at least two, then the point-line truncation of the J-Grassmann geometry coincides with the point-line truncation of the augmented J-Grassmann geometry.

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The structure of the paper is as follows. The remaining subsections of the present section contain most of the necessary definitions and statement of the main result (Theorem 1.1 of Section 1.3). In Section 2 we prove a property of buildings similar to the Deletion Condition for Coxeter groups (Proposition 2.1), and then state some corollaries of this property required in the proof of Theorem 1.1.

Section 3 contains some technical definitions used in the proof of Theorem 1.1, and Section 4 contains the proof of Theorem 1.1. In Section 6 we give two examples of applications of Theorem 1.1. Section 5 contains two results regarding the convex closure of a set of chambers in a building, related to the case $S_p = \emptyset$ of Theorem 1.1 (see Propositions 5.2 and 5.4).

1.1 Definitions of metric properties of graphs and chamber systems. Suppose $G = (V, E)$ is a graph. A walk of length n in G is a sequence of vertices $(x_0, \ldots, x_n)$ such that $\{x_i, x_{i+1}\} \in E$ for all $i \in \{0, \ldots, n-1\}$. The length of the walk w is denoted $\mathrm{l}(w)$. If $w_1 = (x_0, \ldots, x_k)$ and $w_2 = (x_k, \ldots, x_n)$ are walks in G, then we denote $w_1 \circ w_2$ the walk $(x_0, \ldots, x_k, x_{k+1}, \ldots, x_n)$; the walk $w_1 \circ w_2$ is called the *concatenation* of w_1 and w_2. If G is the graph of a chamber system then walks in G are also called *galleries*.

A *geodesic* from x to y in G is a walk from x to y of the smallest possible length. The distance between two vertices $x, y \in V$ is the length of a geodesic from x to y in G. The distance from x to y in G will be denoted $\mathrm{d}_G(x, y)$ or just $\mathrm{d}(x, y)$.

A subset X of V is *convex* in G if, for every $x, y \in X$, the vertices of every geodesic from x to y in G lie in X. A subgraph of G is convex in G if its set of vertices is convex in G.

Let $G = (V, E)$ be a graph, and suppose that $G' = (V', E')$ is a subgraph of G. We say that G' is *strongly gated* in G if, for every $x \in V$, there is a vertex $x' \in V'$ such that, for every $y \in V'$, we have $\mathrm{d}_G(x, y) = \mathrm{d}_G(x, x') + \mathrm{d}_{G'}(x', y)$. The vertex x' is called the *gate* of x in G' and is denoted $\mathrm{gate}_{G'}(x)$. For a subset X of V we let $\mathrm{Gate}_{G'}(X) = \{y \in V' \mid y = \mathrm{gate}_{G'}(x), x \in X\}$.

Suppose $\mathcal{C}$ is a chamber system over the type set I, such that every edge is labelled by a one-element subset of I. Let $w = (c_0, \ldots, c_n)$ be a gallery in $\mathcal{C}$ and suppose that, for every $i \in \{1, \ldots, n\}$, the label of the edge $\{c_{i-1}, c_i\}$ is $\{t_i\}$, $t_i \in I$. Then we say that the *type* of the gallery w is $\mathrm{t}(w) = t_1 \ldots t_n$, a word in the free monoid of words on I.

Suppose $\mathcal{C}$ is a chamber system over a type set I with graph (C, E) and labelling map λ. Suppose $G = (C', E')$ is a subgraph of (C, E). We denote $\mathrm{typ}(G)$ the set of all elements of I that appear in labels of edges of G, that is $\mathrm{typ}(G) = \bigcup_{e \in E'} \lambda(e)$. If X is a set of chambers of $\mathcal{C}$, then $\mathrm{typ}(X)$ denotes $\mathrm{typ}(G)$, where G is the subgraph of $\mathcal{C}$ induced on X. Most of the time we will use the same letter to denote a set of chambers of $\mathcal{C}$, and the chamber subsystem and the subgraph induced on this set in $\mathcal{C}$.

1.2 Buildings and properties of buildings. Let I be a nonempty set and let M be a Coxeter matrix over I. That is, M is a map from $I \times I$ into $\mathbb{N} \cup \{\infty\}$ that takes (i, j) to m_{ij}, such that $m_{ii} = 1$ for all $i \in I$, and $m_{ij} \geq 2$ for all $i, j \in I$ with $i \neq j$. A *chamber system of type* M over I is a connected chamber system in which, for every $\{i, j\} \subseteq I$, all residues of type $\{i, j\}$ are chamber systems of generalized m_{ij}-gons. Note that, in a chamber system of type M, every edge is labelled by a one-element subset of I, and every rank one residue contains at least two chambers.

Let M be a Coxeter matrix over a set I and let $\mathcal{W}$ be the Coxeter group of type M with the set of generators $S = \{s_i \mid i \in I\}$. A *building* of type M over I is a chamber system $\mathcal{B}$ of type M that satisfies one of the following three equivalent conditions.

(P) If two galleries w and w' of reduced types $t_1 \dots t_n$ and $t'_1 \dots t'_{n'}$ have the same initial and terminal chambers, then $s_{t_1} \dots s_{t_n} = s_{t'_1} \dots s_{t'_{n'}}$ in $\mathcal{W}$.
(G) Every gallery of $\mathcal{B}$ of reduced type is a geodesic.
(RG) All residues of $\mathcal{B}$ are strongly gated in $\mathcal{B}$.

Conditions (P) and (G) refer to galleries of reduced type. These are defined as follows. Suppose $\mathcal{C}$ is a chamber system of type M over I. Let $\mathcal{W}$ be a Coxeter group of type M over I with generators $S = \{s_i \mid i \in I\}$. Let w be a gallery in $\mathcal{B}$ of type $t_1 \dots t_n$ and let $r \in \mathcal{W}$ be such that $r = s_{t_1} \dots s_{t_n}$. Then w is a *gallery of reduced type* if $s_{t_1} \dots s_{t_n}$ is a word of the smallest possible length in the free monoid of words on S such that $s_{t_1} \dots s_{t_n} = r$ in $\mathcal{W}$.

Suppose $\mathcal{B}$ is a building of type M over the type set I. Let $\mathcal{W}$ denote the Coxeter group of type M and, also, the Coxeter chamber system of type M. An *apartment* of $\mathcal{B}$ is any map from $\mathcal{W}$ into $\mathcal{B}$ which is an isomorphism of chamber systems over I. The image of $\mathcal{W}$ under this isomorphism is also called an apartment of $\mathcal{B}$ (cf. Proposition 2.1 of [6]). For ease of reference, below we list some properties of buildings. Their proofs can be found in [12], [9], and [10].

(B0) Every pair of chambers of $\mathcal{B}$ is contained in some apartment of $\mathcal{B}$.
(B1) Apartments and residues are convex induced chamber subsystems of $\mathcal{B}$.
(B2) Suppose A and A' are apartments of $\mathcal{B}$ and suppose R_1 and R_2 are residues of $\mathcal{B}$ such that $A \cap R_i \neq \emptyset$ and $A' \cap R_i \neq \emptyset$ for $i = 1, 2$. Then there is an isomorphism of chamber systems over I, $\varphi : A \to A'$, such that, for $i = 1$ and 2, $\varphi(A \cap R_i) = A' \cap R_i$.
(B3) If R is a residue of $\mathcal{B}$ of type J, then R is a building of type $M|J$.
(B4) If $\{Q_\alpha \mid \alpha \in S\}$ is a family of residues of $\mathcal{B}$ and $\bigcap_{\alpha \in S} Q_\alpha \neq \emptyset$, then $\bigcap_{\alpha \in S} Q_\alpha$ is a residue of $\mathcal{B}$ of type $\bigcap_{\alpha \in S} \mathrm{typ}(Q_\alpha)$.
(B5) Suppose R, P, and Q are residues of $\mathcal{B}$ such that $R \cap P \neq \emptyset$, $R \cap Q \neq \emptyset$, and $P \cap Q \neq \emptyset$. Then $R \cap P \cap Q \neq \emptyset$.
(B6) Suppose R is a residue of $\mathcal{B}$, and suppose A_1 is an apartment of R. Then there exists an apartment A of $\mathcal{B}$ such that $A_1 \subseteq A$.

A Coxeter diagram M over I is *spherical* if the Coxeter group of type M is finite. A building $\mathcal{B}$ of type M over I is spherical if M is spherical. If $\mathcal{B}$ is a spherical building, then all residues of $\mathcal{B}$ are spherical buildings.

1.3 Point-line spaces associated with buildings and statement of the main result. An *incidence geometry* over a type set I is a multipartite graph with parts labelled by elements of I. For every $i \in I$, the vertices of the graph belonging to the part labelled i are called the *objects* of the geometry of type i. Two adjacent vertices of the graph are said to be *incident* objects of the geometry. If Γ is an incidence geometry and O is an object of Γ, then we denote $\Gamma(O)$ the set of all objects of Γ incident with O.

A *point-line space* $(\mathcal{P}, \mathcal{L})$ is a set of points $\mathcal{P}$ together with a set $\mathcal{L}$ of subsets of $\mathcal{P}$ such that $|L| \geq 2$ for every $L \in \mathcal{L}$. An incidence geometry over the type set $\{point, line\}$ can be viewed as a point-line space, if every line of the geometry is incident with at least two points and, for every pair of distinct lines, the sets of points incident with them never coincide.

Let $\Sigma = (\mathcal{P}, \mathcal{L})$ be a point-line space. The point-collinearity graph of Σ is a graph $(\mathcal{P}, \mathcal{E})$ with vertex set $\mathcal{P}$ in which two vertices are adjacent if and only if there is a line containing both. A subset S of $\mathcal{P}$ is a *subspace* of Σ if every line that meets X in at least two distinct points is contained in X. A subset S of $\mathcal{P}$ is a *convex subspace* of Σ if S is a subspace of Σ and S is convex in $(\mathcal{P}, \mathcal{E})$.

Let $\Sigma = (\mathcal{P}, \mathcal{L})$ be a point-line space. Suppose S is a subset of $\mathcal{P}$. We denote $\langle S \rangle_\Sigma$ or $\langle S \rangle$ the intersection of all subspaces of Σ containing S, and we say that $\langle S \rangle$ is the *span* or the *subspace closure* of S in Σ. We denote $\langle\langle S \rangle\rangle_\Sigma$ or $\langle\langle S \rangle\rangle$ the intersection of all convex subspaces of $(\mathcal{P}, \mathcal{L})$ containing S, and we say that $\langle\langle S \rangle\rangle$ is the *convex subspace closure* of S in $(\mathcal{P}, \mathcal{L})$.

Let $\mathcal{B}$ be a building of type M over I, and let D be the diagram graph of M and $\mathcal{B}$. That is, D is a graph with vertex set I in which two distinct vertices $i, j \in I$ are adjacent if and only if $m_{ij} \geq 3$. We now define the augmented J-Grassmann geometry of $\mathcal{B}$.

Let $S_p \subseteq I$ and let $I'' = \{k \in \mathbb{N} \mid 1 \leq k \leq |I'|\}$, where I' is the union of vertex sets of all connected components of D that meet $I - S_p$. Let Γ be the $(I - S_p)$-Grassmann geometry of $\mathcal{B}$ defined as in [6, 7]. Then Γ is a geometry over the type set I''. For an object O of Γ we denote R_O the residue of $\mathcal{B}$ corresponding to O.

The *augmented* $(I - S_p)$*-Grassmann geometry of* $\mathcal{B}$ is a geometry Γ' over the type set $I''' = I'' \cup \{\infty\}$, obtained from Γ by adding an object O of type ∞ incident with all objects of Γ. The residue R_O of $\mathcal{B}$ corresponding to the object O is the entire building $\mathcal{B}$.

Both, the J-Grassmann geometry and the augmented J-Grassmann geometry, can be constructed for any chamber system $\mathcal{C}$, not necessarily a building. We just define the diagram graph of $\mathcal{C}$ as a graph with vertex set I, the type set of $\mathcal{C}$, in which two vertices $i, j \in I$ are adjacent if and only if at least one residue of $\mathcal{C}$ of type $\{i, j\}$ is not a generalized digon.

Let $\mathcal{B}$ be a building of type M over I with diagram graph D. Let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$. Then the set of points $\mathcal{P}$ is in bijective correspondence with the set of all residues of $\mathcal{B}$ of type S_p, and the set of lines $\mathcal{L}$ is in bijective correspondence with the set of all residues of $\mathcal{B}$ of all possible types T, such that $T = \{i\} \cup (S_p - (D_{0,1}(i) \cap S_p))$ for some $i \in I - S_p$, (we denote $D_{0,1}(i)$ the set of all vertices of the graph D at distance 0 or 1 from i). A point $p \in \mathcal{P}$ and a line $L \in \mathcal{L}$ are incident in Σ if and only if $\mathrm{R}_p \cap \mathrm{R}_L \neq \emptyset$. Every rank one residue of $\mathcal{B}$ is contained in a residue corresponding to a point or a line of Σ.

For every building $\mathcal{B}$ and every set S_p, the set of points of the augmented $(I - S_p)$-Grassmann geometry Γ' is nonempty, and the point-collinearity graph of Γ' is connected. This makes the augmented J-Grassmann geometry more convenient for the purposes of the present paper than the J-Grassmann geometry. If $|I'| \geq 2$, then the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$ coincides with the point-line truncation of the $(I - S_p)$-Grassmann geometry of $\mathcal{B}$.

Let $\mathcal{B}$ be a building of type M over I with diagram graph D, and suppose that $\Sigma = (\mathcal{P}, \mathcal{L})$ is the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$. Let X be a set of chambers of $\mathcal{B}$. We denote $\mathcal{P}_X$ the set of all points $p \in \mathcal{P}$ such that $\mathrm{R}_p \cap X \neq \emptyset$, and we say that $\mathcal{P}_X$ is the *point shadow* of X.

Our goal in the present paper is to prove the following theorem. For the definition of the opposition involution see Section 2.2 (or see [9]).

Theorem 1.1. *Let $\mathcal{B}$ be a spherical building with Coxeter diagram M over I. Let $S_p \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$. Assume that at least one of the following two conditions holds.*

(C1) $|I - S_p| = 1$.

(C2) *The set $I - S_p$ is stabilized by the automorphism of the diagram M induced by the opposition involution of the Coxeter chamber system of type M (this includes the case $S_p = I$).*

Let A be an apartment of $\mathcal{B}$. Then $\langle\langle \mathcal{P}_A \rangle\rangle = \mathcal{P}$.

If the building $\mathcal{B}$ is not assumed to be spherical then the conclusion of Theorem 1.1 fails. This can be seen in the following example. Let $\mathcal{B}$ be a generalized m-gon with $m = \infty$ over the type set $I = \{i, j\}$, and let $S_p = \emptyset$. Let Σ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$, and let $\mathcal{G}$ denote the point-collinearity graph of Σ. The points of Σ are the chambers of $\mathcal{B}$, the lines of Σ are the panels of $\mathcal{B}$, and the graph $\mathcal{G}$ is the graph of the chamber system $\mathcal{B}$. Let A be an apartment of $\mathcal{B}$. Then A is a doubly infinite gallery with edges labelled alternately by $\{i\}$ and $\{j\}$, consisting of pairwise distinct chambers. We have $\mathcal{P}_A = A$, and $\langle\langle \mathcal{P}_A \rangle\rangle$ is the union of the sets of chambers of the panels of $\mathcal{B}$ that meet A. Therefore, if $\mathcal{B}$ is not thin, then $\langle\langle \mathcal{P}_A \rangle\rangle$ is not the entire building $\mathcal{B}$ in this case.

No example of a spherical building, satisfying at least one of the conditions (C1) and (C2) such that the conclusion of Theorem 1.1 fails, is known to the author.

When the diagram graph of the building has more than one connected component, Theorem 1.1 can be combined with Proposition 1.2 below. Proposition 3.2, referred to in the proof of Proposition 1.2, is just Proposition 3.6 of [7] restated for augmented Grassmann geometries (see Section 3).

Proposition 1.2. *Let $\mathcal{B}$ be a building with Coxeter diagram M over I and diagram graph D. Let $S_p \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$.*

Let $I_1, \ldots, I_k$ be the vertex sets of the connected components of the diagram graph D, where k is a positive integer. For every $i \in \{1, \ldots, k\}$, for every residue R_i of $\mathcal{B}$ of type I_i, and for every apartment A_i of R_i, suppose that $\langle\langle (\mathcal{P}_i)_{A_i} \rangle\rangle_{\Sigma_i} = \mathcal{P}_i$, where $\Sigma_i = (\mathcal{P}_i, \mathcal{L}_i)$ is the point-line truncation of the $(I_i - (I_i \cap S_p))$-Grassmann geometry of R_i.

Let A be an apartment of $\mathcal{B}$. Then $\langle\langle \mathcal{P}_A \rangle\rangle = \mathcal{P}$.

Proof. Let A be an apartment of $\mathcal{B}$. Let c be a chamber of A. For every $i \in \{1, \ldots, k\}$, let A_i be the residue of A of type I_i containing c. The chamber system A is the product of the chamber systems A_i.

For every $i \in \{1, \ldots, k\}$, let R_i be the residue of $\mathcal{B}$ of type I_i containing A_i, and let $\Sigma_i = (\mathcal{P}_i, \mathcal{L}_i)$ be the point-line truncation of the $(I_i - (I_i \cap S_p))$-Grassmann geometry of R_i.

Let $\varphi_i : \mathcal{P}_i \to \mathcal{P}$ be the map that takes a point or line $O \in \mathcal{P}_i \cup \mathcal{L}_i$ to the point or line $O' \in \mathcal{P} \cup \mathcal{L}$ such that $\mathrm{R}_{O'} \cap R_i$ is the residue of R_i corresponding to O. By Proposition 3.2 φ_i is an isomorphism of Σ_i onto $\Sigma|\mathcal{P}_{R_i}$, where $\Sigma|\mathcal{P}_{R_i} = (\mathcal{P}_{R_i}, \mathcal{L}|\mathcal{P}_{R_i})$ and $\mathcal{L}|\mathcal{P}_{R_i}$ is the set of lines of Σ meeting $\mathcal{P}_{R_i}$ in at least two distinct points.

By Theorem 6.1 of [6] $\mathcal{P}_{R_i}$ is a convex subspace of Σ, therefore the convex subspace closure in $\Sigma|\mathcal{P}_{R_i}$ coincides with the convex subspace closure inside Σ. Since by hypothesis $\langle\!\langle (\mathcal{P}_i)_{A_i} \rangle\!\rangle_{\Sigma_i} = \mathcal{P}_i$, we obtain that $\langle\!\langle \mathcal{P}_{A_i} \rangle\!\rangle = \mathcal{P}_{R_i}$.

The above shows that, for every $i \in \{1, \ldots, k\}$, $\mathcal{P}_{R_i} \subseteq \langle\!\langle \mathcal{P}_A \rangle\!\rangle$. We claim that this implies that $\langle\!\langle \mathcal{P}_A \rangle\!\rangle = \mathcal{P}$. It suffices to consider the case $k = 2$ since the other cases follow from this case by induction on k.

Suppose $k = 2$. Then Σ is the product geometry $\Sigma = \Sigma|\mathcal{P}_{R_1} \times \Sigma|\mathcal{P}_{R_2} \cong \Sigma_1 \times \Sigma_2$. Let $p \in \mathcal{P}$. Let Q_1 and Q_2 be the residues of $\mathcal{B}$ of types I_1 and I_2 intersecting R_p, and let $\Sigma_1' = \Sigma|\mathcal{P}_{Q_1}$ and $\Sigma_2' = \Sigma|\mathcal{P}_{Q_2}$. We have $\Sigma_1' \cong \Sigma_1$, $\Sigma_2' \cong \Sigma_2$, and $\Sigma = \Sigma|\mathcal{P}_{Q_1} \times \Sigma|\mathcal{P}_{Q_2} \cong \Sigma_1' \times \Sigma_2'$.

Let p_1 and p_2 be the projections of p onto $\mathcal{P}_{R_1}$ and $\mathcal{P}_{R_2}$. That is, $\{p_1\} = \mathcal{P}_{R_1} \cap \mathcal{P}_{Q_2}$ and $\{p_2\} = \mathcal{P}_{R_2} \cap \mathcal{P}_{Q_1}$. Any geodesic from p_1 to p_2 in $\mathcal{G}$ projects onto a walk from p_1 to p in $\mathcal{P}_{Q_2}$ and onto a walk from p to p_2 in $\mathcal{P}_{Q_1}$. For every edge of $\mathcal{G}$, one projection is an edge and the other projection is a single vertex. Therefore $\mathrm{d}_{\mathcal{G}}(p_1, p_2) \geq \mathrm{d}_{\mathcal{G}}(p_1, p) + \mathrm{d}_{\mathcal{G}}(p, p_2)$. This shows that p lies on a geodesic from p_1 to p_2 in $\mathcal{G}$. Since $p_1, p_2 \in \langle\!\langle \mathcal{P}_A \rangle\!\rangle$, we have $p \in \langle\!\langle \mathcal{P}_A \rangle\!\rangle$.

We showed that every point of $\mathcal{P}$ is in $\langle\!\langle \mathcal{P}_A \rangle\!\rangle$. Therefore, $\langle\!\langle \mathcal{P}_A \rangle\!\rangle = \mathcal{P}$. □

2 A property of buildings and its corollaries

The main purpose of this section is to prove Proposition 2.6 of Section 2.2, that will be used in the proof of Theorem 1.1. The proof of Proposition 2.6 depends on Proposition 2.1 to construct a geodesic of a certain type between two chambers of a building $\mathcal{B}$. If the building $\mathcal{B}$ is thin, that is if $\mathcal{B}$ is a Coxeter chamber system, then Proposition 2.1 becomes the Deletion Condition for Coxeter groups (the statement of the Deletion Condition for Coxeter groups can be found in [5]).

2.1 A property of buildings. The proof of Proposition 2.1 uses the following property of Coxeter chamber systems. Suppose $\mathcal{C}$ is a Coxeter chamber system over a type set I. Let x and y be chambers of $\mathcal{C}$ and let W be a geodesic from x to y in $\mathcal{C}$. Suppose r is a reflection in the Coxeter group of $\mathcal{C}$. That is, r is an automorphism of the chamber system $\mathcal{C}$ that interchanges two vertices of some edge of $\mathcal{C}$. Then there are two possibilities: (1) r stabilizes no edge of W or (2) r stabilizes exactly one edge of W. In Case (1) the reflection r does not stabilize any edge of any geodesic from x to y in $\mathcal{C}$, and in Case (2) the reflection r stabilizes exactly one edge of every geodesic from x to y in $\mathcal{C}$. For a proof of this property of Coxeter groups see [9].

Proposition 2.1. *Let $\mathcal{B}$ be a building of type M over a type set I. Let x and y be chambers of $\mathcal{B}$, and let W be a gallery from x to y in $\mathcal{B}$. Suppose that the type of W is $t_1 \dots t_k$, where $t_1, \dots, t_k \in I$. If W is not a geodesic from x to y in $\mathcal{B}$, then there is a gallery W' from x to y of type $t_1 \dots \widehat{t_j} \dots t_k$, or of type $t_1 \dots \widehat{t_i} \dots \widehat{t_j} \dots t_k$, where $i, j \in \{1, \dots, k\}$ and $\widehat{t_\alpha}$ indicates that the element t_α is omitted.*

Proof. Suppose that $W = (c_0, \dots, c_k)$ is a gallery from $c_0 = x$ to $c_k = y$ in $\mathcal{B}$ of type $t_1 \dots t_k$, and suppose that W is not a geodesic. Let c_{j+1} be the first chamber of W such that $\mathrm{d}(x, c_{j+1}) \neq \mathrm{d}(x, c_j) + 1$. Note that $1 \leq j \leq k-1$ and $c_j \neq x, y$.

Let W_1 be the part of the gallery W beginning with x and ending with c_j, and let W_2 be the part of W beginning with c_{j+1} and ending with y. Then $W = W_1 \circ (c_j, c_{j+1}) \circ W_2$, and by the choice of c_{j+1} the gallery W_1 is a geodesic from x to c_j.

Case 1. Suppose first that $\mathrm{d}(x, c_{j+1}) = \mathrm{d}(x, c_j)$. Let Q be the panel of $\mathcal{B}$ of type $\{t_j\}$ containing c_j and c_{j+1}. Let A_j be an apartment of $\mathcal{B}$ containing x and c_j, let A_{j+1} be an apartment of $\mathcal{B}$ containing x and c_{j+1}, and let $g = \mathrm{gate}_Q(x)$. Apartments A_j and A_{j+1} exist by (B0).

Since by (B1) A_j and A_{j+1} are convex in $\mathcal{B}$, we have $A_j \cap Q = \{c_j, g\}$ and $A_{j+1} \cap Q = \{c_{j+1}, g\}$. By (B2) there is an isomorphism $\varphi : A_j \to A_{j+1}$ such that $\varphi(x) = x$, $\varphi(g) = g$, and $\varphi(c_j) = c_{j+1}$.

By convexity of A_j (property (B1)), all vertices of W_1 lie in A_j. Let W_1' be the gallery which is the image of W_1 under φ. Then $W_1' \circ W_2$ is a gallery from x to y in $\mathcal{B}$. The type of W_1' is $t_1 \dots t_{j-1}$ and the type of W_2 is $t_{j+1} \dots t_k$, therefore $\mathrm{t}(W_1' \circ W_2) = t_1 \dots t_{j-1} t_{j+1} \dots t_k$.

Case 2. Suppose that $\mathrm{d}(x, c_{j+1}) = \mathrm{d}(x, c_j) - 1$. Let A be an apartment of $\mathcal{B}$ containing x and c_j. Let W_1' be a geodesic from x to c_{j+1} in $\mathcal{B}$. The galleries W_1 and $W_1' \circ (c_{j+1}, c_j)$ are both geodesics from x to c_j in $\mathcal{B}$. Since A is convex in $\mathcal{B}$, and $x, c_j \in A$, all the chambers of both W_1 and $W_1' \circ (c_{j+1}, c_j)$ lie in A.

Let r be the reflection in the Coxeter group of A such that $r(c_j) = c_{j+1}$. Since r stabilizes the edge $\{c_j, c_{j+1}\}$ of $W_1' \circ (c_{j+1}, c_j)$, it must stabilize exactly one edge of W_1.

Let c_i and c_{i+1} be the chambers of W_1 such that $r(c_i) = c_{i+1}$. Let U be the part of W_1 beginning with x and ending with c_i, and let V be the part of W_1 beginning with c_{i+1} and ending with c_j. Then $W_1 = U \circ (c_i, c_{i+1}) \circ V$.

Let V' be the image of V under r. Then V' begins with c_i and ends with c_{j+1}. Let $W' = U \circ V' \circ W_2$. Then W' is a gallery from x to y in $\mathcal{B}$. We have $\mathrm{t}(V') = \mathrm{t}(V)$ and $\mathrm{t}(W') = \mathrm{t}(U)\,\mathrm{t}(V')\,\mathrm{t}(W_2)$. Therefore $\mathrm{t}(W') = t_1 \dots t_{i-1} t_{i+1} \dots t_{j-1} t_{j+1} \dots t_k$. □

The proof of Case 2 in the above proposition is essentially a proof of the Deletion Condition for Coxeter groups (cf. Exercise 4, p. 24, of [9]) and can be replaced with a reference to the Exchange Condition or the Deletion Condition.

The following corollary is immediate from Proposition 2.1.

Corollary 2.2. *Let $\mathcal{B}$ be a building of type M over a type set I. Let x and y be chambers of $\mathcal{B}$, and suppose W is a gallery from x to y in $\mathcal{B}$. If W is not a geodesic, then there is a geodesic W' from x to y in $\mathcal{B}$ such that* $\mathrm{t}(W')$ *is obtained from* $\mathrm{t}(W)$ *by omitting one or more terms.*

2.2 Geodesics between two points in $(\mathcal{P}, \mathcal{L})$. In this section we are going to use Proposition 2.1, together with Lemma 2.5, to show existence of certain geodesics in the point-collinearity graph of the augmented J-Grassmann geometry of a spherical building. The result is stated as Proposition 2.6.

First, we prove Proposition 2.3 and Corollary 2.4, that apply to all buildings, not necessarily spherical. Corollary 2.4 shows that, if $\mathcal{C}$ is a convex induced chamber subsystem of a building $\mathcal{B}$, and Σ is the $(I - S_p)$-Grassmann geometry of $\mathcal{B}$, then the point-collinearity graph of the J-Grassmann geometry of $\mathcal{C}$, corresponding to Σ, embeds into the point-collinearity graph of Σ isometrically.

We need the following notation. Let $\mathcal{B}$ be a building of type M over I. Let $S_p \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$. Let $\mathcal{G}$ denote the point-collinearity graph of Σ.

Since every residue of $\mathcal{B}$ of rank 1 is contained in a residue corresponding to a point or line of Σ, every gallery $W = (c_0, \ldots, c_n)$ of $\mathcal{B}$ determines a walk $w = (p_0, \ldots, p_k)$ in the graph $\mathcal{G}$, that consists of the points corresponding to the residues of $\mathcal{B}$ of type S_p traversed by the gallery W. We denote this walk $\mathrm{w}_{\mathcal{G}}(W)$, and we assume that no two consecutive points of $\mathrm{w}_{\mathcal{G}}(W)$ are equal to each other.

Conversely, suppose $w = (p_0, \ldots, p_n)$ is a walk in $\mathcal{G}$. Let $x \in \mathrm{R}_{p_0}$ and let $y \in \mathrm{R}_{p_n}$. Every point residue is a connected chamber system. Also, by Lemma 3.10 of [7], if $\langle p, q \rangle$ is a line of Σ, then there is a chamber in R_p connected by an edge to a chamber in R_q. Combining these two facts we obtain that there is a gallery W from x to y in $\mathcal{B}$ such that $\mathrm{w}_{\mathcal{G}}(W) = w$.

Proposition 2.3. *Let $\mathcal{B}$ be a building of type M over I. Let $S_p \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$. Suppose $p, q \in \mathcal{P}$. Let $x \in \mathrm{R}_p$ and let $y \in \mathrm{R}_q$. Then there is a geodesic W from x to y in $\mathcal{B}$, such that $\mathrm{w}_{\mathcal{G}}(W)$ is a geodesic from p to q in $\mathcal{G}$.*

Proof. Let $w = (p_0, \ldots, p_n)$ be a geodesic from $p = p_0$ to $q = p_n$ in $\mathcal{G}$. Let W be a gallery in $\mathcal{B}$ beginning with x and ending with y, such that $\mathrm{w}_{\mathcal{G}}(W) = w$.

Suppose that W is not a geodesic in $\mathcal{B}$. By Corollary 2.2, there is a geodesic W' from x to y in $\mathcal{B}$, such that $\mathrm{t}(W')$ can be obtained from $\mathrm{t}(W)$ by omitting one or more terms. Let $w' = \mathrm{w}_{\mathcal{G}}(W')$. Then $\mathrm{l}(w') \leq \mathrm{l}(w)$. Therefore w' is a geodesic from p to q in $\mathcal{G}$. □

The following is an immediate corollary of Proposition 2.3.

Corollary 2.4. *Let $\mathcal{B}$ be a building of type M over I. Let $S_p \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$. Let $\mathcal{G}$ denote the point-collinearity graph of Σ.*

Suppose C is a convex set of chambers in $\mathcal{B}$, and let $\mathcal{C}$ be the chamber subsystem of $\mathcal{B}$ induced on C. Let $\Sigma(\mathcal{C}) = (\mathcal{P}(\mathcal{C}), \mathcal{L}(\mathcal{C}))$ be the point-line truncation of the augmented $(T - (T \cap S_p))$-Grassmann geometry of $\mathcal{C}$, where $T = \mathrm{typ}(C)$. Let $\mathcal{G}(\mathcal{C})$ denote the point-collinearity graph of $\Sigma(\mathcal{C})$.

Let φ be the map that takes a point $p \in \mathcal{P}_C$ to the point $p' \in \mathcal{P}(\mathcal{C})$ corresponding to the residue $\mathrm{R}_p \cap C$ of $\mathcal{C}$. Then, for all $p, q \in \mathcal{P}_C$ we have $\mathrm{d}_{\mathcal{G}}(p, q) = \mathrm{d}_{\mathcal{G}(\mathcal{C})}(\varphi(p), \varphi(q))$.

Let $\mathcal{B}$ be a spherical building of type M over I. Then the graph of the chamber system $\mathcal{B}$ has finite diameter. Let d denote the diameter of $\mathcal{B}$. Two chambers c and c' of $\mathcal{B}$ are *opposite* in $\mathcal{B}$ if $\mathrm{d}(c, c') = d$.

Let $\mathcal{C}$ be a Coxeter chamber system of type M. Then the diameter of $\mathcal{C}$ is d, and every chamber of $\mathcal{C}$ has a unique opposite in $\mathcal{C}$. The map $\mathrm{op}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that takes every chamber of $\mathcal{C}$ to its opposite in $\mathcal{C}$ is, in general, a non-type preserving automorphism of the chamber system $\mathcal{C}$. That is, $\mathrm{op}_{\mathcal{C}}$ is an automorphism of the graph of the chamber system $\mathcal{C}$ which, for every $i \in I$, maps the edges of $\mathcal{C}$ labelled $\{i\}$ to the edges of $\mathcal{C}$ labelled $\{\mathrm{op}_M(i)\}$, where op_M denotes the automorphism of the diagram M induced by $\mathrm{op}_{\mathcal{C}}$. The map $\mathrm{op}_{\mathcal{C}}$ is called the *opposition involution* of $\mathcal{C}$. Suppose Q is a residue of $\mathcal{C}$. Then the image of Q under $\mathrm{op}_{\mathcal{C}}$ is a residue of $\mathcal{C}$ and is called the *opposite* residue of Q in $\mathcal{C}$.

Suppose Q and Q' are residues of the building $\mathcal{B}$. Then Q and Q' are *opposite* in $\mathcal{B}$ if there is an apartment A of $\mathcal{B}$ such that $Q \cap A$ and $Q' \cap A$ are opposite in A. If Q and Q' are opposite residues of $\mathcal{B}$, then they are opposite in every apartment of $\mathcal{B}$ meeting both. These definitions and proofs of the facts used in them can be found in [11] or [9].

Let $\mathcal{B}$ be a spherical building and let d denote the diameter of $\mathcal{B}$. Suppose Q and Q' are residues of $\mathcal{B}$. Then Q and Q' are opposite in $\mathcal{B}$ if and only if the map of Q into Q' defined by $x \mapsto \mathrm{gate}_{Q'}(x)$ is a bijection of Q onto Q', and, for every $c \in Q$, $\mathrm{d}_{\mathcal{B}}(c, \mathrm{gate}_{Q'}(c)) = d - d_{Q'}$, where $d_{Q'}$ denotes the diameter of the residue Q'.

Suppose c and c' are opposite chambers of $\mathcal{B}$, and suppose Q is a residue of $\mathcal{B}$ containing c. Then there is a unique residue Q' of $\mathcal{B}$ that contains c' and is opposite to Q in $\mathcal{B}$.

Lemma 2.5. *Let $\mathcal{C}$ be a spherical building of type M over I. Let $S_p \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{C}$. Let $\mathcal{G}$ be the point-collinearity graph of Σ.*

Suppose that $\mathcal{C}$ is thin, and suppose that the set $I - S_p$ is stabilized by op_M, the automorphism of M induced by the opposition involution of $\mathcal{C}$.

Let $p, q \in \mathcal{P}$ be such that the residues R_p and R_q are opposite in $\mathcal{C}$. Then, for every $\alpha \in I - S_p$, there is a geodesic $w = (p_0, p_1, \ldots, p_n)$ from $p = p_0$ to $q = p_n$ in $\mathcal{G}$, such that R_p and R_{p_1} are connected by an edge labelled $\{\alpha\}$.

Proof. Let $\mathrm{op}_{\mathcal{C}}$ be the opposition involution of $\mathcal{C}$. By the hypothesis the set S_p is stabilized by op_M, therefore the image under $\mathrm{op}_{\mathcal{C}}$ of a residue of $\mathcal{C}$ of type S_p is a residue of $\mathcal{C}$ of type S_p. In particular, there exist points p and q as in the hypothesis.

We can assume that $I - S_p \neq \emptyset$. Therefore, $p \neq q$. Let $x \in \mathrm{R}_p$ and let $z = \mathrm{op}_{\mathcal{C}}(x)$. Then $z \in \mathrm{R}_q$. By Proposition 2.3 there is a geodesic U from x to z in $\mathcal{C}$, such that $\mathrm{w}_{\mathcal{G}}(U)$ is a geodesic from p to q in $\mathcal{G}$. Let $u = \mathrm{w}_{\mathcal{G}}(U)$ and suppose that $u = (p_0, p_1, \ldots, p_n)$, where $p_0 = p$ and $p_n = q$.

Let V be the image of U under $\mathrm{op}_{\mathcal{C}}$. Since x and z are opposite in $\mathcal{C}$, the gallery V is a gallery from z to x.

Let $v = \mathrm{w}_{\mathcal{G}}(V)$ and suppose that $v = (q_0, \ldots, q_m)$. By hypothesis op_M stabilizes the set S_p. Therefore $q_0 = q$ and $q_m = p$. Furthermore, $m = n$ and, for every $i \in \{0, \ldots, n\}$, R_{q_i} is the residue of $\mathcal{C}$ opposite to R_{p_i}.

For $i \in \{0, \ldots, n\}$, let φ_i be an automorphism of $\mathcal{C}$ taking R_{p_i} to R_p. The automorphism φ_i exists since the automorphism group of $\mathcal{C}$ is transitive on the set of chambers of $\mathcal{C}$. Let ψ_i be the automorphism of $\mathcal{G}$ induced by φ_i.

Let $u_i = (p_i, p_{i+1}, \ldots, q, q_1, \ldots, q_i)$. Since R_{q_i} is the unique residue of $\mathcal{C}$ opposite to R_{p_i}, and R_q is the unique residue of $\mathcal{C}$ opposite to R_p, and $\psi_i(p_i) = p$, we have $\psi_i(q_i) = q$. Therefore ψ_i maps the walk u_i to a walk u_i' from p to q. Since $\mathrm{l}(u_i) = \mathrm{l}(u) = n$, the walk u_i' is a geodesic from p to q in $\mathcal{G}$.

Let $\alpha \in I - S_p$. Since the chambers x and z are opposite in $\mathcal{C}$, the type of the gallery U represents the longest word in the Coxeter group corresponding to $\mathcal{C}$. This implies that every element of I occurs as the label of an edge of U at least once. Therefore, there is $i \in \{0, \ldots, n-1\}$ such that R_{p_i} and $\mathrm{R}_{p_{i+1}}$ are connected in U by an edge labelled $\{\alpha\}$. Then the walk u_i', constructed in the preceding paragraph, is a geodesic from p to q, and R_p and $\mathrm{R}_{\psi_i(p_{i+1})}$ are connected by an edge labelled $\{\alpha\}$. □

Proposition 2.6. *Let $\mathcal{B}$ be a spherical building of type M over I. Let $S_p \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$. Let $\mathcal{G}$ be the point-collinearity graph of Σ.*

Let $p, q \in \mathcal{P}$ and suppose $p \neq q$. Assume that at least one of the following conditions holds.

(1) $|I - S_p| = 1$.

(2) *The set $I - S_p$ is stabilized by op_M, and the residues R_p and R_q are opposite in $\mathcal{B}$.*

Then, for every $\alpha \in I - S_p$, there is a geodesic $w = (p_0, p_1, \ldots, p_n)$ from $p = p_0$ to $q = p_n$ in $\mathcal{G}$, such that R_p and R_{p_1} are connected by an edge labelled $\{\alpha\}$.

Proof. Suppose Part (1) of the hypothesis holds. Let $\alpha \in I - S_p$, and denote by $w = (p_0, p_1, \ldots, p_n)$ a geodesic from $p_0 = p$ to $p_n = q$ in $\mathcal{G}$. By hypothesis $|I - S_p| = 1$, therefore $I - S_p = \{\alpha\}$. Since p and p_1 are distinct collinear points of Σ, by Lemma 3.10 of [7] the residues R_p and R_{p_1} are connected by an edge, and the label of this edge is not in S_p. Therefore, the edge is labelled $\{\alpha\}$, and w is the required geodesic.

Suppose Part (2) of the hypothesis holds. Let $\alpha \in I - S_p$. Let A be an apartment of $\mathcal{B}$ that intersects both R_p and R_q. Let $\Sigma(A) = (\mathcal{P}(A), \mathcal{L}(A))$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of A, and let $\mathcal{G}(A)$ be the point-collinearity graph of $\Sigma(A)$.

Let φ be the map that takes a point or line O of $\Sigma(A)$ to the point or line O' of Σ such that $\mathrm{R}_{O'} \cap A$ is the residue of A corresponding to O. By Proposition 3.6 of [7] (see Section 3, Proposition 3.2) the map φ exists and is an isomorphism of $\Sigma(A)$ into Σ that maps $\mathcal{P}(A)$ onto $\mathcal{P}_A$.

We have $p, q \in \mathcal{P}_A$. Let v be a geodesic from $\varphi^{-1}(p)$ to $\varphi^{-1}(q)$ in $\mathcal{G}(A)$. Let $w = (p_0, \ldots, p_n)$ be the image of v under φ. We have $p_0 = p$, $p_n = q$. Since A is convex in $\mathcal{B}$, by Corollary 2.4 w is a geodesic from p to q in $\mathcal{G}$. By Lemma 2.5 we can choose the geodesic v so that R_p and R_{p_1} are connected by an edge labelled $\{\alpha\}$. □

3 Additional definitions

Let M be a Coxeter matrix over a set I, and let $\mathcal{B}$ be a building of type M. We denote by D the diagram graph of M and $\mathcal{B}$. Let $S_p \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$.

First, we recall some notation introduced in [6]. Let $T \subseteq I$. Then $\mathrm{K}_{S_p}(T)$ denotes the union of the vertex sets of all connected components of the graph $D|T$ that meet $I - S_p$. If Q is a residue of $\mathcal{B}$ of type T, and Q' is a residue of Q of type $\mathrm{K}_{S_p}(T)$, then $\mathcal{P}_Q = \mathcal{P}_{Q'}$.

Suppose $\mathcal{C}$ is a chamber subsystem of the building $\mathcal{B}$, and suppose that the type set of $\mathcal{C}$ is T. Then we denote $\Sigma(\mathcal{C}) = (\mathcal{P}(\mathcal{C}), \mathcal{L}(\mathcal{C}))$ the point-line truncation of the augmented $(T - (T \cap S_p))$-Grassmann geometry of $\mathcal{C}$ (this differs from the notation in [7], where $\Sigma(\mathcal{C})$ denotes the point-line truncation of $(T - (T \cap S_p))$-Grassmann geometry of $\mathcal{C}$).

Suppose that X is a subset of $\mathcal{P}$. We denote by $\mathcal{L}|X$ the set of all lines of Σ that meet X in at least two distinct points. We let $\Sigma|X$ be the point-line geometry whose set of points is X, whose set of lines is $\mathcal{L}|X$, and whose incidence is inherited from Σ.

Let $\mathrm{cham}_{\mathcal{B}} : 2^{\mathcal{P}} \to 2^{\mathcal{B}}$ be the map defined by the rule that, if X is a subset of $\mathcal{P}$, then $\mathrm{cham}_{\mathcal{B}}(X) = \bigcup_{p \in X} \mathrm{R}_p$. The set $\mathrm{cham}_{\mathcal{B}}(X)$ can be viewed as the inverse image of X under the map $2^{\mathcal{B}} \to 2^{\mathcal{P}}$ that takes a set of chambers Y of $\mathcal{B}$ to its point shadow $\mathcal{P}_Y$. For a subset $\mathcal{P}'$ of $\mathcal{P}$, we have $\mathcal{P}_{\mathrm{cham}_{\mathcal{B}}(\mathcal{P}')} = \mathcal{P}'$.

Let X be a subset of $\mathcal{P}$. We denote by $\widetilde{X}$ the set of points $\bigcup_{L \in \mathcal{L}|X} \mathcal{P}_L$, where $\mathcal{P}_L$ denotes the set of all points of Γ incident with the line L. That is, $\widetilde{X}$ consists of all points of Γ incident in Γ with lines of $\Sigma|X$. We have $X \subseteq \widetilde{X}$.

Suppose C is a set of chambers of $\mathcal{B}$. Then we let $\widetilde{C}$ denote the set $\mathrm{cham}_{\mathcal{B}}(\widetilde{\mathcal{P}_C})$. We have $\mathcal{P}_C \subseteq \widetilde{\mathcal{P}_C}$ and $C \subseteq \widetilde{C}$. If C is an apartment of $\mathcal{B}$, then Lemma 3.1 below shows that $\widetilde{C}$ contains every panel of $\mathcal{B}$ on every chamber of C.

Lemma 3.1. *Let $\mathcal{B}$ be a building of type M over I, let $S_p \subseteq I$, and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$.*

Suppose A is an apartment of $\mathcal{B}$. Then, for every chamber $c \in A$, the set $\widetilde{A}$ contains all panels of $\mathcal{B}$ on c.

Proof. Let $c \in A$, let $i \in I$, and let Q be a panel of $\mathcal{B}$ on c of type $\{i\}$. Suppose first that $i \in S_p$. Let $p \in \mathcal{P}_A$ be such that $c \in \mathrm{R}_p$. Then $Q \subseteq \mathrm{R}_p$ and $\mathrm{R}_p \subseteq \widetilde{A}$, therefore $Q \subseteq \widetilde{A}$.

Suppose $i \notin S_p$. Let $T = \{i\} \cup (S_p - (D_{0,1}(i) \cap S_p))$. Then residues of type T of $\mathcal{B}$ correspond to lines of Σ. Let $L \in \mathcal{L}$ be the line corresponding to the residue of type T containing Q, and let $\mathcal{P}_L$ denote the set of all points of Σ incident with L. A point $q \in \mathcal{P}$ is incident in Σ with the line L if and only if $\mathrm{R}_q \cap Q \neq \emptyset$. Therefore $\mathcal{P}_L = \mathcal{P}_Q$. Moreover, if $q \in \mathcal{P}_L$, then by (B4) $\mathrm{R}_q \cap Q$ is a single chamber.

Since $Q \cap A$ is a panel of A, the line L is incident with two distinct points of $\mathcal{P}_A$. Therefore $L \in \mathcal{P}_A$ and $\mathcal{P}_L \subseteq \widetilde{\mathcal{P}_A}$. This implies $Q \subseteq \mathrm{cham}_{\mathcal{B}}(\mathcal{P}_L) \subseteq \widetilde{A}$. □

We close this section with the following observation. Let $\mathcal{B}$ be a building over a type set I. Suppose $S_p \subseteq I$ is such that $|\mathrm{K}_{S_p}(I)| = 1$. Then the set of lines of the

$(I-S_p)$-Grassmann geometry of $\mathcal{B}$ is empty but the set of lines of the augmented $(I-S_p)$-Grassmann geometry of $\mathcal{B}$ is nonempty (it consists of exactly one line, whose corresponding residue is the building $\mathcal{B}$ itself). Similarly, if $S_p = I$ (that is $|\mathrm{K}_{S_p}(I)| = 0$), then the set of points of the $(I-S_p)$-Grassmann geometry of $\mathcal{B}$ is empty but the set of points of the augmented $(I-S_p)$-Grassmann geometry of $\mathcal{B}$ is nonempty and consists of exactly one point, corresponding to the building $\mathcal{B}$.

The absence of points or lines in the $(T-(T\cap S_p))$-Grassmann geometry of the chamber subsystem $\mathcal{C}$ when $|\mathrm{K}_{S_p}(T)| \leq 1$ was the only reason why we required $|\mathrm{K}_{S_p}(T)| \geq 2$ in Propositions 3.5 and 3.6 of [7]. Therefore, if in the statement of Proposition 3.6 of [7] we replace the $(I-S_p)$- and $(T-(T\cap S_p))$-Grassmann geometries with the corresponding augmented Grassmann geometries, then we can drop the requirement $|\mathrm{K}_{S_p}(T)| \geq 2$ and obtain the following.

Proposition 3.2. *Suppose $\mathcal{B}$ is a building of type M over I. Let $S_p \subseteq I$, let Γ be the augmented $(I-S_p)$-Grassmann geometry of $\mathcal{B}$, and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of Γ.*

Let $T \subseteq I$. Let $\mathcal{C}$ be a chamber subsystem of $\mathcal{B}$, and suppose $\mathcal{C}$ is a chamber system of type $M|T$. Let C denote the set of chambers of $\mathcal{C}$. Then $\Sigma|\mathcal{P}_C \cong \Sigma(\mathcal{C})$, where the isomorphism takes a point or line O of $\Sigma|\mathcal{P}_C$ to the point or line of $\Sigma(\mathcal{C})$ corresponding to the residue $\mathrm{R}_O \cap C$ of $\mathcal{C}$.

4 Proof of Theorem 1.1

First we prove Lemma 4.1, which is the main part of the proof of Theorem 1.1. For some motivations for the proof see Section 5.

We need the notion of retraction onto an apartment in a building. Suppose $\mathcal{B}$ is a building of type M over I. Let A be an apartment of $\mathcal{B}$ and let c be a chamber of A. The *retraction* of $\mathcal{B}$ onto A with center c is a map $\mathrm{retr}_{A,c} : \mathcal{B} \to A$ defined as follows. Let x be a chamber of $\mathcal{B}$, and let w be a geodesic from c to x in $\mathcal{B}$. Let w' be the unique gallery in A whose initial chamber is c and whose type is $\mathrm{t}(w)$. Then $\mathrm{retr}_{A,c}(x)$ is defined to be the terminal chamber of w'. By property (P) of buildings $\mathrm{retr}_{A,c}(x)$ does not depend on the choice of the geodesic w from c to x. The map $\mathrm{retr}_{A,c} : \mathcal{B} \to A$ is a morphism of chamber systems over I, and its restriction to A is the identity map.

Suppose now that $\mathcal{B}$ is a spherical building. Let A be an apartment of $\mathcal{B}$ and let c be a chamber of A. Suppose x is a chamber of $\mathcal{B}$, and let galleries w and w' be as in the preceding paragraph. Let x' be the unique chamber of A opposite to $\mathrm{retr}_{A,c}(x)$. Then c lies on a geodesic $w'' \circ w'$ in A from x' to $\mathrm{retr}_{A,c}(x)$. The gallery $w'' \circ w'$ is of reduced type, therefore the gallery $w'' \circ w$ is of reduced type, and by property (G) of buildings the gallery $w'' \circ w$ is a geodesic from x' to x in $\mathcal{B}$. In particular, x' is opposite to x in $\mathcal{B}$. Details can be found in Section 4.2 of [11].

Lemma 4.1. *Let $\mathcal{B}$ be a building of rank at least 2, and suppose that the hypothesis of Theorem 1.1 holds. If only condition (C1) holds and $\mathrm{rank}(\mathcal{B}) \geq 3$, then assume that the implication of Theorem 1.1 is true for all residues of $\mathcal{B}$ different from $\mathcal{B}$.*

Let A be an apartment of $\mathcal{B}$. Then $\langle\langle \mathcal{P}_A \rangle\rangle = \mathcal{P}$.

Proof. Let $X = \mathrm{cham}_{\mathcal{B}}(\langle\langle \mathcal{P}_A \rangle\rangle)$. The lemma will be proved if we can show that X is the set of all chambers of $\mathcal{B}$ or, equivalently, if we can prove the following statement.

(S) *For every chamber x of $\mathcal{B}$, the chamber x and all the panels of $\mathcal{B}$ on x are in X.*

First, we prove statement (S′).

(S′) *If $y \in A$, then all panels of $\mathcal{B}$ on y are contained in X.*

Let the set $\widetilde{A}$ be defined as in Section 3. Then $\widetilde{A} \subseteq X$. By Lemma 3.1 the set $\widetilde{A}$ contains all panels of $\mathcal{B}$ on every chamber of A, therefore so does X. This proves (S′).

Let $c \in A$ and let x be a chamber of $\mathcal{B}$. To prove statement (S) we use induction on $\mathrm{d}(c, x)$. If $\mathrm{d}(c, x) = 0$, then $x = c$ and by (S′) all panels of $\mathcal{B}$ on x lie in X.

Suppose $\mathrm{d}(c, x) \geq 1$ and assume that, for every chamber y of $\mathcal{B}$ at distance $\mathrm{d}(c, x) - 1$ from c, the chamber y and all the panels of $\mathcal{B}$ on y lie in X. Let a be a chamber of $\mathcal{B}$ adjacent to x and such that $\mathrm{d}(c, a) = \mathrm{d}(c, x) - 1$. By the induction hypothesis $a \in X$, and all panels of $\mathcal{B}$ on a are contained in X. Therefore $x \in X$.

It remains to show that all panels of $\mathcal{B}$ on x are contained in X. Let $\alpha \in I$ and let Q_α be the panel of type $\{\alpha\}$ on x. We need to show that $Q_\alpha \subseteq X$.

Let $p \in \mathcal{P}$ be such that $x \in \mathrm{R}_p$. We consider two cases separately.

Case 1. $\alpha \in S_p$. In this case $Q_\alpha \subseteq \mathrm{R}_p$. Since the set X is a union of residues of $\mathcal{B}$ of type S_p, $x \in X$, and R_p is the unique residue of type S_p containing x, we have $\mathrm{R}_p \subseteq X$. Therefore $Q_\alpha \subseteq X$.

Case 2. $\alpha \in I - S_p$. Let x' be the unique chamber of A opposite to $\mathrm{retr}_{A,c}(x)$ in A. Then x' is opposite to x in $\mathcal{B}$. Let Q be the unique residue of $\mathcal{B}$ containing x' and opposite to R_p in $\mathcal{B}$. Let $x'' = \mathrm{gate}_Q(x)$, and let $q \in \mathcal{P}$ be such that $x'' \in \mathrm{R}_q$.

First, we are going to show that $q \in \langle\langle \mathcal{P}_A \rangle\rangle$ (step (2.1)). Then we use Proposition 2.6 to obtain a geodesic W from x to x'' in $\mathcal{B}$ with certain properties (step (2.2)). Then we use the geodesic W to show that $Q_\alpha \subseteq X$.

(2.1) $\mathcal{P}_Q \subseteq \langle\langle \mathcal{P}_A \rangle\rangle$. *In particular, $q \in \langle\langle \mathcal{P}_A \rangle\rangle$.*

Suppose first that (C2) holds. Then the sets S_p and $I - S_p$ are stabilized by op_M, therefore $\mathrm{typ}(Q) = S_p$ and $Q = \mathrm{R}_q$. We remark that this includes, in particular, the case $S_p = \emptyset$, that is the case when the points of Γ are the chambers of $\mathcal{B}$, and the lines of Γ are the panels of $\mathcal{B}$. Since $Q \cap A \neq \emptyset$, we have $q \in \mathcal{P}_A$.

Suppose now that (C1) holds. That is, $|I - S_p| = 1$. If $\mathrm{rank}(\mathcal{B}) = 2$, then $|S_p| = 1$, and R_p and Q are panels of $\mathcal{B}$. Since $x' \in Q \cap A$, by (S′) $Q \subseteq X$. Therefore $\mathcal{P}_Q \subseteq \mathcal{P}_X$.

Suppose $\mathrm{rank}(\mathcal{B}) \geq 3$. Let $T = \mathrm{typ}(Q)$. By (B3) Q is a building of type $M|T$. Let $\Delta = \Sigma|\mathcal{P}_Q = (\mathcal{P}_Q, \mathcal{L}|\mathcal{P}_Q)$ be the geometry of points and lines of Σ lying entirely in $\mathcal{P}_Q$, and let $\Delta' = \Sigma(Q) = (\mathcal{P}(Q), \mathcal{L}(Q))$ be the point-line truncation of the augmented $(T - (T \cap S_p))$-Grassmann geometry of Q. By Proposition 3.2 $\Delta \cong \Delta'$, where the isomorphism, which we denote φ, takes a point or line O of $\Sigma|\mathcal{P}_Q$ to the point or line of $\Sigma(Q)$ corresponding to the residue $\mathrm{R}_O \cap Q$ of Q.

We have $\mathrm{typ}(Q) = T = \mathrm{op}_M(S_p)$. Since $S_p \neq I$, the residue Q is not the entire building $\mathcal{B}$. Since the set S_p satisfies condition (C1), that is $|I - S_p| = 1$, we have $|T - (T \cap S_p)| \leq 1$. Therefore, the set $T \cap S_p$ either satisfies condition (C1) with respect to the diagram $M|T$, or $T \cap S_p = T$, that is $T \cap S_p$ satisfies (C2) with respect to the diagram $M|T$. The intersection $Q \cap A$ is an apartment of Q (see Lemma 3.4 of [7] or

Lemma 2.4 of [6]), therefore by hypothesis $\langle\!\langle (\mathcal{P}(Q))_{Q\cap A} \rangle\!\rangle_{\Delta'} = \mathcal{P}(Q)$. Applying the isomorphism φ^{-1} to $\Sigma(Q)$ we obtain that $\langle\!\langle \mathcal{P}_{Q\cap A} \rangle\!\rangle_{\Delta} = \mathcal{P}_Q$.

By Corollary 6.2 of [6] (or by Proposition 2.6 of [7]) the set $\mathcal{P}_Q$ is a convex subspace of Σ. Therefore, the convex subspace closure of $\mathcal{P}_{Q\cap A}$ in Δ coincides with the convex subspace closure of $\mathcal{P}_{Q\cap A}$ in Σ. That is, $\mathcal{P}_Q = \langle\!\langle \mathcal{P}_{Q\cap A} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{P}_A \rangle\!\rangle$. This completes the proof of (2.1).

(2.2) *There is a geodesic W from x to x'' in $\mathcal{B}$, such that the first edge of W is labelled $\{\alpha\}$, and all vertices of* $\mathrm{w}_{\mathcal{G}}(W)$ *are in $\langle\!\langle \mathcal{P}_A \rangle\!\rangle$.*

By the current hypothesis $I - S_p \neq \emptyset$. That is $S_p \neq I$, therefore $\mathrm{R}_p \cap Q = \emptyset$ and $p \neq q$. By Proposition 2.6 there is a geodesic $z = (q_0, q_1, \dots, q_n)$ from $p = q_0$ to $q = q_n$ in $\mathcal{G}$, such that R_p and R_{q_1} are connected by an edge labelled $\{\alpha\}$.

Let Z be a gallery in $\mathcal{B}$ beginning at x and ending at x'', such that $\mathrm{w}_{\mathcal{G}}(Z) = z$. By Corollary 2.2 there is a geodesic W from x to x'' such that either $\mathrm{t}(W) = \mathrm{t}(Z)$ or $\mathrm{t}(W)$ is obtained from $\mathrm{t}(Z)$ by omitting one or more terms.

Let $w = \mathrm{w}_{\mathcal{G}}(W)$ and suppose that $w = (p_0, \dots, p_m)$, where $p_0 = p$ and $p_m = q$. The walk z is a geodesic from p to q in $\mathcal{G}$, and $\mathrm{l}(w) \leq \mathrm{l}(z)$. Therefore $m = n$, and no letters belonging to $I - S_p$ were omitted when switching from Z to W. In particular, the residues R_p and R_{p_1} are connected by an edge labelled $\{\alpha\}$.

Let x_1 be the chamber that follows x in the geodesic W. Since by its definition $x'' = \mathrm{gate}_{\mathrm{R}_q}(x)$, and the residues Q and R_p are opposite in $\mathcal{B}$, we have $x = \mathrm{gate}_{\mathrm{R}_p}(x'')$. The gallery W is a geodesic from x to x'' in $\mathcal{B}$, therefore $x_1 \notin \mathrm{R}_p$. It follows that $x_1 \in \mathrm{R}_{p_1}$, and $\{x, x_1\}$ is an edge labelled $\{\alpha\}$.

Since $x \in X \cap \mathrm{R}_p$, we have $p \in \langle\!\langle \mathcal{P}_A \rangle\!\rangle$. Also, by (2.1) $q \in \langle\!\langle \mathcal{P}_A \rangle\!\rangle$. The set $\langle\!\langle \mathcal{P}_A \rangle\!\rangle$ is convex in $\mathcal{G}$, therefore all vertices of the geodesic w lie in $\langle\!\langle \mathcal{P}_A \rangle\!\rangle$. This completes the proof of (2.2).

Let the geodesics W and w be as in 2.2. Let $L = \langle p, p_1 \rangle$. By Lemma 3.10 of [7] $x, x_1 \in \mathrm{R}_L$ and $\alpha \in \mathrm{typ}(\mathrm{R}_L)$. Therefore $Q_\alpha \cap \mathrm{R}_L \neq \emptyset$ and $\mathrm{typ}(Q_\alpha) \subseteq \mathrm{typ}(\mathrm{R}_L)$. By property (B4) this implies that $Q_\alpha \subseteq \mathrm{R}_L$.

The set $\langle\!\langle \mathcal{P}_A \rangle\!\rangle$ is a subspace of Σ, and the line L has two points p and p_1 in $\langle\!\langle \mathcal{P}_A \rangle\!\rangle$. Therefore $\mathcal{P}_L \subseteq \langle\!\langle \mathcal{P}_A \rangle\!\rangle$, where $\mathcal{P}_L$ denotes the set of all points of Σ incident with L. Since $X = \mathrm{cham}_{\mathcal{B}}(\langle\!\langle \mathcal{P}_A \rangle\!\rangle)$, this implies $\mathrm{cham}_{\mathcal{B}}(\mathcal{P}_L) \subseteq X$. Therefore $\mathrm{R}_L \subseteq X$ and $Q_\alpha \subseteq X$. This completes the proof of Case 2 and the proof of the lemma. □

Proof of Theorem 1.1. Assume that the hypothesis of Theorem 1.1 holds. In particular, $\mathcal{B}$ is a spherical building, A is an apartment of $\mathcal{B}$, and the set S_p satisfies at least one of the conditions (C1) and (C2). We need to show that $\langle\!\langle \mathcal{P}_A \rangle\!\rangle = \mathcal{P}$.

Observe that if $\mathrm{rank}(\mathcal{B}) = 0$ (that is, if $\mathcal{B}$ is a single chamber and Σ is a single point), or if $\mathrm{rank}(\mathcal{B}) = 1$ (that is, $\mathcal{B}$ is a single panel and Σ is either a single point or a single line), then $\langle\!\langle \mathcal{P}_A \rangle\!\rangle = \mathcal{P}$. Therefore, in the rest of the proof we assume that $\mathrm{rank}(\mathcal{B}) \geq 2$. We consider two cases.

(1) $|I - S_p| = 1$, that is condition (C1) holds. To show that $\langle\!\langle \mathcal{P}_A \rangle\!\rangle = \mathcal{P}$ we use induction on the rank of $\mathcal{B}$, starting with $\mathrm{rank}(\mathcal{B}) = 2$. The proofs of both, the initial induction step and the general induction step, are immediate from Lemma 4.1.

(2) The set $I - S_p$ is stabilized by op_M, that is condition (C2) holds. Then $\langle\!\langle \mathcal{P}_A \rangle\!\rangle = \mathcal{P}$ by Lemma 4.1. □

5 Some remarks on the case $S_p = \emptyset$

As has been mentioned before, if $\mathcal{B}$ is a building (or any camber system) over a type set I, and $S_p = \emptyset$, then the points of the $(I - S_p)$-Grassmann geometry of $\mathcal{B}$ are the chambers of $\mathcal{B}$ and the lines are the panels of $\mathcal{B}$. In this section we describe two results, Propositions 5.2 and Proposition 5.4, that relate to this case and that were the initial motivation for our proof of Theorem 1.1. For some other results pertaining to convex sets of chambers in a building see [1].

The proof of Lemma 4.1, Case 2, is reminiscent of the proof of Theorem 4.1.1 of [11]. Proposition 5.2 below can easily be proved using the same induction approach, and using Lemma 5.1. Combining Proposition 5.2 and Lemma 3.1 one immediately obtains Corollary 5.3, which is a special case of Theorem 1.1 corresponding to $S_p = \emptyset$, a special case of condition (C2).

Making an additional assumption that $\mathcal{B}$ is thick and repeating almost exactly the proof of Theorem 4.1.1 of [11] one obtains Proposition 5.4.

Lemma 5.1. *Let $\mathcal{B}$ be a spherical building of type M over I, and let X be a convex set of chambers of $\mathcal{B}$. Let Q and Q' be opposite panels of $\mathcal{B}$. Suppose that $Q \subseteq X$ and $Q' \cap X \neq \emptyset$. Then $Q' \subseteq X$.*

Proposition 5.2. *Let $\mathcal{B}$ be a spherical building of type M over I and let A be an apartment of $\mathcal{B}$. Suppose that X is a convex set of chambers of $\mathcal{B}$ containing A and such that, for every chamber $x \in A$, all panels of $\mathcal{B}$ on x are contained in X. Then $X = \mathcal{B}$.*

Corollary 5.3. *Let $\mathcal{B}$ be a spherical building of type M over I, and let A be an apartment of $\mathcal{B}$. Suppose $S_p = \emptyset$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$. Then $\langle\!\langle \mathcal{P}_A \rangle\!\rangle = \mathcal{P}$.*

Proof. Since $S_p = \emptyset$, the points of Σ are the chambers of $\mathcal{B}$, the lines of Σ are the panels of $\mathcal{B}$, and the point-collinearity graph of Σ is the graph of the chamber system $\mathcal{B}$. Therefore $\mathcal{P}_A = A$ and $\widetilde{\mathcal{P}_A} = \widetilde{A}$. This implies $\widetilde{A} \subseteq \langle \mathcal{P}_A \rangle \subseteq \langle\!\langle \mathcal{P}_A \rangle\!\rangle$. By Lemma 3.1 the set $\widetilde{A}$ contains A and all the panels of B on every chamber of A, therefore so does the set $\langle\!\langle \mathcal{P}_A \rangle\!\rangle$. Since the set $\langle\!\langle \mathcal{P}_A \rangle\!\rangle$ is convex in $\mathcal{B}$, by Proposition 5.2 $\langle\!\langle \mathcal{P}_A \rangle\!\rangle = \mathcal{B}$. □

Proposition 5.4. *Let $\mathcal{B}$ be a thick spherical building of type M over I. Suppose A is an apartment of $\mathcal{B}$, and let c be a chamber of A. Suppose that X is a convex set of chambers of $\mathcal{B}$, containing A and all the panels of $\mathcal{B}$ on c. Then $X = \mathcal{B}$.*

Proof. Let x be a chamber of $\mathcal{B}$. We use induction on $\mathrm{d}(x, c)$ to show that x and all the panels of $\mathcal{B}$ on x are contained in $\mathcal{B}$.

If $x = c$, then the statement is the hypothesis of the proposition. Suppose $\mathrm{d}(x, c) \geq 1$. Let a be a chamber of $\mathcal{B}$ adjacent to x and such that $\mathrm{d}(c, a) = \mathrm{d}(c, x) - 1$. Then x is contained in a panel of $\mathcal{B}$ on a, therefore by the induction hypothesis $x \in X$. It remains to show that all the panels of $\mathcal{B}$ on x are contained in X.

Let Q_1 be the panel of $\mathcal{B}$ on a and x, and let Q_2 be any panel of $\mathcal{B}$ on x. We can assume that $Q_2 \neq Q_1$, since otherwise $Q_2 \subseteq X$ by the induction hypothesis. Let a' and

x' be the unique chambers of A opposite to $\mathrm{retr}_{A,c}(a)$ and $\mathrm{retr}_{A,c}(x)$ respectively. Let Q_1' be the panel of $\mathcal{B}$ on a' and x'. Then Q_1 and Q_1' are opposite in $\mathcal{B}$.

Since $\mathcal{B}$ is thick, there is a chamber $x'' \in Q_1'$ opposite to both a and x. Since $a' \in A$, and by the induction hypothesis $Q_1 \subseteq X$, by Lemma 5.1 applied to Q_1 and Q_1' we have $Q_1' \subseteq X$. Therefore $x'' \in X$.

Let Q_2' be the panel of $\mathcal{B}$ on x'' opposite to Q_2, and let Q_2'' be the panel of $\mathcal{B}$ on a opposite to Q_2'. By the induction hypothesis $Q_2'' \subseteq X$, therefore by Lemma 5.1 applied to Q_2'' and Q_2' we obtain that $Q_2' \subseteq X$. The chamber x is in X, and the panel Q_2 is opposite to Q_2' in $\mathcal{B}$, therefore applying Lemma 5.1 to Q_2' and Q_2 we obtain that $Q_2 \subseteq X$.

We have shown that, for every chamber x of $\mathcal{B}$, the chamber x and all the panels of $\mathcal{B}$ on x are contained in X. Therefore $X = \mathcal{B}$. □

6 Applications

In this section we describe two applications of Theorem 1.1. In Section 6.1 we characterize, under certain conditions, those convex subspaces of the augmented J-Grassmann geometry of a building $\mathcal{B}$ that are isomorphic to shadows of spherical residues of $\mathcal{B}$ (Theorem 6.2 and Corollaries 6.3 and 6.4). In Section 6.2 we show that, if Σ is the point-line truncation of the augmented J-Grassmann geometry Γ of a building, then the point shadow X of a plane of Γ cannot be properly contained in a subspace which is a generalized polygon, provided that the point-collinearity graph of $\Sigma|X$ has finite diameter (Proposition 6.6).

6.1 A characterization of convex subspaces isomorphic to the shadow of a spherical residue. Suppose $\mathcal{B}$ is a building of type M over a type set I. Let $J \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented J-Grassmann geometry of $\mathcal{B}$. Theorem 1.1 can be used together with Theorem 3.3 of [6] to characterize convex subspaces of Σ isomorphic to the subspace $\Sigma|\mathcal{P}_R$, where R is a spherical residue of $\mathcal{B}$. We formulate this result below as Theorem 6.2 (see also Corollaries 6.3 and 6.4). Theorem 6.2 will be proved using the approach used to prove Theorems 2.15, 3.6, and 4.7 of [3]. A different characterization of point shadows of residues of $\mathcal{B}$, similar to Theorem 3.3 of [6], will be given in a forthcoming paper.

To state and prove Theorem 6.2 we need the following notion. Suppose $(\mathcal{P}, \mathcal{L})$ is a point-line space with the set of points $\mathcal{P}$ and the set of lines $\mathcal{L}$. Suppose that all singular subspaces of $(\mathcal{P}, \mathcal{L})$ are projective spaces. Let $(\mathcal{P}, \mathcal{E})$ be the point collinearity graph of $(\mathcal{P}, \mathcal{L})$. We say that a subset X of $\mathcal{P}$ is *singularly independent* or *s-independent* in $(\mathcal{P}, \mathcal{L})$, if every finite subset C of X such that the graph $(\mathcal{P}, \mathcal{E})|C$ is a clique spans a singular subspace of $(\mathcal{P}, \mathcal{L})$ of projective dimension $|C| - 1$. We say that an induced subgraph G of $(\mathcal{P}, \mathcal{E})$ is s-independent, if its vertex set is s-independent.

Suppose now that $\Sigma = (\mathcal{P}, \mathcal{L})$ is the point-line truncation of an augmented J-Grassmann geometry of a building $\mathcal{B}$. By Corollary 3.14 of [7] every singular subspace of Σ is a projective space, therefore the notion of s-independence can be used in Σ. The following lemma shows that the point shadow of an apartment of $\mathcal{B}$ is always s-independent in Σ.

Lemma 6.1. *Let $\mathcal{B}$ be a building of type M over I. Let $S_p \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$. Suppose A is an apartment of $\mathcal{B}$. Then the set $\mathcal{P}_A$ is s-independent in Σ.*

Proof. If $|\mathrm{K}_{S_p}(I)| = 0$, then Σ is a single point and the statement is obvious. Suppose that $|\mathrm{K}_{S_p}(I)| \geq 1$. Let $\mathcal{G}$ be the point-collinearity graph of Σ, and let C be a clique of $\mathcal{G}$ contained in $\mathcal{P}_A$. Suppose that $|C|$ is finite. We need to show that the span of C in Σ is a singular space of projective rank $|C| - 1$.

By Proposition 3.2 the map $\varphi : \Sigma|\mathcal{P}_A \to \Sigma(A)$, that takes a point or line O of $\Sigma|\mathcal{P}_A$ to the point or line of $\Sigma(A)$ corresponding to the residue $\mathrm{R}_O \cap A$ of A, is an isomorphism of geometries. Let C' be the image of C under φ.

Every line of the geometry $\Sigma(A)$ is incident with exactly two points, therefore the clique C' is a subspace of $\Sigma(A)$. Since C' is a singular subspace of $\Sigma(A)$ of finite projective rank $|C| - 1$, by Corollary 3.15 of [7] there is a residue A' of A of type T, for some $T \subseteq I$, such that $C' = \mathcal{P}(A)_{A'}$. Moreover, the diagram $M|\mathrm{K}_{S_p}(T)$ is of type $\mathsf{A}_{|C'|-1}$, and $(I - S_p) \cap \mathrm{K}_{S_p}(T)$ is one of its end nodes.

Let Q be the residue of $\mathcal{B}$ of type T containing A'. Let Q' be a residue of Q of type $\mathrm{K}_{S_p}(T)$. Using (B4) and (B5) we see that $\Sigma(Q) \cong \Sigma(Q')$. Therefore, the geometry $\Sigma(Q)$ is the point-line geometry of a building of type $\mathsf{A}_{|C'|-1}$, corresponding to an end node of the diagram. By 6.3 of [11] $\Sigma(Q)$ is a projective space of projective rank $|C'| - 1$.

By Proposition 3.2 the map $\psi : \Sigma|\mathcal{P}_Q \to \Sigma(Q)$, that takes a point or line O of $\Sigma|\mathcal{P}_Q$ to the point or line of $\Sigma(Q)$ corresponding to the residue $Q \cap \mathrm{R}_O$ of Q, is an isomorphism of geometries. By Proposition 2.6 of [7] $\mathcal{P}_Q$ is a subspace of Σ, therefore $\Sigma|\mathcal{P}_Q$ is a singular subspace of Σ of projective rank $|C'| - 1$.

By Lemma 2.4 of [6] (or by Lemma 3.4 of [7]) A' is an apartment of Q. Therefore, the set $\mathcal{P}(Q)_{A'}$ spans the projective space $\Sigma(Q)$. Since $\mathcal{P}_{A'} = \psi^{-1}(\mathcal{P}(Q)_{A'})$, the set $\mathcal{P}_{A'}$ spans $\Sigma|\mathcal{P}_Q$. The clique C is contained in $\mathcal{P}_A$ and $\varphi(C) = C' = \mathcal{P}(A)_{A'}$, therefore $C = \mathcal{P}_{A'}$ and C spans $\Sigma|\mathcal{P}_Q$. □

Theorem 6.2. *Suppose that $\mathcal{B}$ is a building of type M over a type set I. Let $S_p \subseteq I$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$. Let $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ be the point-collinearity graph of Σ.*

Suppose T is a subset of I such that the Coxeter diagram $M|\mathrm{K}_{S_p}(T)$ is spherical. Let R be a residue of $\mathcal{B}$ of type T, and let A be an apartment of R. Suppose that the following condition holds.

(Apt)$_T$ *For every s-independent induced subgraph $G = (V, E)$ of $\mathcal{G}$ isomorphic to the induced subgraph $\mathcal{G}|\mathcal{P}_A$, there is an apartment A' of a residue of $\mathcal{B}$ of type T such that $V = \mathcal{P}_{A'}$.*

Then, for every convex subspace $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ of Σ isomorphic to the subspace $\Sigma|\mathcal{P}_R$, there is a residue Q of $\mathcal{B}$ of type T such that $\mathcal{P}_1 = \mathcal{P}_Q$ and $\Sigma_1 = \Sigma|\mathcal{P}_Q$.

Proof. Let T, R, and A be as in the hypothesis. Let $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ be a convex subspace of Σ and suppose $\varphi : \mathcal{P}_R \to \mathcal{P}_1$ is an isomorphism of the subspace $\Sigma|\mathcal{P}_R$ onto the subspace Σ_1. Let $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{E}_1)$ denote the point-collinearity graph of Σ_1.

Let $V = \varphi(\mathcal{P}_A)$, and let $G = (V, E)$ be the subgraph of $\mathcal{G}_1$ induced on V. By Lemma 6.1 $\mathcal{P}(R)_A$ is s-independent in $\Sigma(R)$, and by Proposition 3.2 $\Sigma|\mathcal{P}_R \cong \Sigma(R)$.

Therefore $\mathcal{P}_A$ is s-independent in $\Sigma|\mathcal{P}_R$. Since φ is an isomorphism, the set V is s-independent in Σ_1. Since Σ_1 is a subspace of Σ, the set V is s-independent in Σ. Therefore, by condition (Apt)$_T$ there is a residue Q of $\mathcal{B}$ of type T, and there is an apartment A' of Q, such that $V = \mathcal{P}_{A'}$ and $G = \mathcal{G}|\mathcal{P}_{A'}$.

We claim that the convex subspace closure of V in the space $\Sigma|\mathcal{P}_Q$ is $\mathcal{P}_Q$. Let Q' be a residue of Q of type $\mathrm{K}_{S_p}(T)$, such that $Q' \cap A' \neq \emptyset$. Then by (B4) and (B5) $\Sigma(Q) \cong \Sigma(Q')$, where the isomorphism takes a point or line O of $\Sigma(Q)$, corresponding to a residue X of Q, to the point or line of $\Sigma(Q')$ corresponding to the residue $Q' \cap X$.

By Proposition 3.2 $\Sigma|\mathcal{P}_Q \cong \Sigma(Q)$, therefore $\Sigma|\mathcal{P}_Q \cong \Sigma(Q')$, and V is the image in $\Sigma|\mathcal{P}_Q$ of the point shadow of the apartment $Q' \cap A'$ of Q'. By hypothesis the residue Q' is spherical, therefore by Theorem 1.1 the convex subspace closure of V in the space $\Sigma|\mathcal{P}_Q$ is $\mathcal{P}_Q$. This proves the claim.

An argument similar to the argument used to prove the preceding claim shows that the convex subspace closure of V in the space Σ_1 is $\mathcal{P}_1$.

The set $\mathcal{P}_1$ is a convex subspace of Σ by hypothesis, and the set $\mathcal{P}_Q$ is a convex subspace of Σ by Corollary 6.2 of [6]. Therefore, $\mathcal{P}_1$ and $\mathcal{P}_Q$ are both equal to the convex subspace closure of V in Σ. Hence, $\mathcal{P}_1 = \mathcal{P}_Q$. □

As was mentioned in the proof of Theorem 6.2, by Corollary 6.2 of [6] the point shadow of a residue of a building $\mathcal{B}$ is always a convex subspace of the J-Grassmann geometry of $\mathcal{B}$.

Below we describe two specific instances of Theorem 6.2.

Example 1. Geometries from buildings with diagram $\mathsf{Y}_{m,l,n}$. Let $\mathcal{B}$ be a building with Coxeter diagram M of type $\mathsf{Y}_{m,l,n}$, where $m, n > 0$ and $l \geq 0$. Assume that the three arms of the diagram are labelled $(-m, \ldots, -1, 0)$, $(0, 1', \ldots, l')$, and $(0, 1, \ldots, n)$, where 0 is the label of the branching node and $|\{1', \ldots, l'\}| = l$. If $\mathcal{B}$ is spherical, then $\mathcal{B}$ has Coxeter diagram A_k, D_k, or E_k, where if the diagram of $\mathcal{B}$ is E_k then $k = 6, 7$ or 8. If $l = 0$, then $\mathcal{B}$ has Coxeter diagram A_k with $k = m + 1 + n$.

Let I denote the type set of $\mathcal{B}$. Let $S_p = I - \{l'\}$, and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $\{l'\}$-Grassmann geometry of $\mathcal{B}$. Let $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ denote the point-collinearity graph of Σ.

Let $\mathcal{T}$ be the collection of all subsets T of I, such that $M|\,\mathrm{K}_{S_p}(T)$ is spherical and T has one of the following forms: $I - \{\gamma\}$, $I - \{\alpha\}$, $I - \{\beta\}$, $I - \{\alpha, \beta\}$, where $\gamma \in \{0, 1', \ldots, l'\}$, $\alpha \in \{-m, \ldots, -1\}$, and $\beta \in \{1, \ldots, n\}$.

Corollary 6.3. *Suppose $\mathcal{B}$ is a building with Coxeter diagram $\mathsf{Y}_{m,l,n}$, where $m, n > 0$ and $l \geq 0$, and let $\Sigma = (\mathcal{P}, \mathcal{L})$ and $\mathcal{T}$ be as described above.*

Let $T \in \mathcal{T}$, let R be a residue of $\mathcal{B}$ of type T, and let $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ be a convex subspace of Σ isomorphic to the subspace $\Sigma|\mathcal{P}_R$. Then there is a residue Q of $\mathcal{B}$ of type T such that $\mathcal{P}_1 = \mathcal{P}_Q$ and $\Sigma_1 = \Sigma|\mathcal{P}_Q$.

Proof. By Theorem 4.5 of [7] condition (Apt)$_T$ is satisfied for every $T \in \mathcal{T}$, therefore the conclusion follows from Theorem 6.2. □

Example 2. Geometries $A_{n,\{1,n\}}$. Let n be a positive integer, $n \geq 2$, and let $\mathcal{B}$ be a building of type A_n. Assume that the consecutive nodes of the diagram are labelled $1, \ldots, n$, and let $I = \{1, \ldots, n\}$. Let $S_p = I - \{1, n\}$, and let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the point-line truncation of the augmented $\{1, n\}$-Grassmann geometry of $\mathcal{B}$. Let $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ denote the point-collinearity graph of Σ.

Let $\mathcal{T}$ be the collection of all subsets of I of the forms $I - \{\gamma\}$ and $I - \{\alpha, \beta\}$, where $1 \leq \gamma \leq n$ and $1 \leq \alpha < \beta \leq n$. As a corollary of Theorem 6.2, and Theorem 4.9 of [7], we obtain Corollary 6.4 below. Note, that in Corollary 6.4 we do not require the subspace $\mathcal{P}_1$ to be convex in Σ.

Corollary 6.4. *Suppose $\mathcal{B}$ is a building with Coxeter diagram A_n, $n \geq 2$, and let the geometry $\Sigma = (\mathcal{P}, \mathcal{L})$ and the collection $\mathcal{T}$ be as described above.*

Let $T \in \mathcal{T}$, let R be a residue of $\mathcal{B}$ of type T, and let $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ be a subspace of Σ isomorphic to the subspace $\Sigma|\mathcal{P}_R$. Then there is a residue Q of $\mathcal{B}$ of type T such that $\mathcal{P}_1 = \mathcal{P}_Q$ and $\Sigma_1 = \Sigma|\mathcal{P}_Q$.

Proof. Let $T \in \mathcal{T}$, and suppose $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ is a subspace of Σ isomorphic to the subspace $\Sigma|\mathcal{P}_R$, where R is a residue of $\mathcal{B}$ of type T. First we show that Σ_1 is convex in Σ.

Let $\mathcal{G}' = (\mathcal{P}_1, \mathcal{E}_1)$ be the point-collinearity graph of Σ_1. The space Σ_1 is a projective space or a product of two projective spaces, therefore the graph $\mathcal{G}'$ is a grid and its diameter is at most 2. Suppose that $x, y \in \mathcal{P}_1$ and $\mathrm{d}_{\mathcal{G}'}(x, y) = 2$. To show that Σ_1 is convex in Σ it suffices to show that the vertices of all geodesics from x to y in $\mathcal{G}$ lie in $\mathcal{G}'$.

Since Σ_1 is a subspace of Σ and $\mathrm{d}_{\mathcal{G}'}(x, y) \neq 1$, we have $\mathrm{d}_{\mathcal{G}}(x, y) \neq 1$. Therefore $\mathrm{d}_{\mathcal{G}}(x, y) = 2$. Since $\mathcal{G}'$ is a grid, $|\mathcal{G}'(x) \cap \mathcal{G}'(y)| = 2$. For any $a, b \in \mathcal{P}$ with $\mathrm{d}_{\mathcal{G}}(a, b) = 2$, we have $|\mathcal{G}(a) \cap \mathcal{G}(b)| = 1$ or 2, therefore $\mathcal{G}(x) \cap \mathcal{G}(y) \subseteq \mathcal{G}'(x) \cap \mathcal{G}'(y)$ and the vertices of all geodesics from x to y in $\mathcal{G}$ lie in $\mathcal{G}'$. Therefore Σ_1 is convex in Σ.

By Theorem 4.9 of [7] condition $(\mathrm{Apt})_T$ is satisfied, therefore the conclusion follows from Theorem 6.2. □

Corollary 6.4 can be proved directly, without using Theorem 6.2.

6.2 An application to subspaces containing planes. A generalized n-gon is a bipartite graph of diameter n and girth $2n$, in which every vertex lies on at least two edges. If $n = \infty$, then a generalized n-gon is a tree without end vertices. A generalized n-gon can be viewed as a point-line geometry and, if $n \geq 3$, then a generalized n-gon can be viewed as a point-line space. The following lemma shows that a proper convex subspace of a generalized n-gon with $n \geq 3$ cannot be a generalized polygon of finite diameter.

Lemma 6.5. *Let $\Sigma = (\mathcal{P}, \mathcal{L})$ be the geometry of points and lines of a generalized n-gon with $n \geq 3$. Suppose $\mathcal{P}_1$ is a convex subspace of Σ, and let $\Sigma_1 = (\mathcal{P}_1, \mathcal{L}_1)$ denote the geometry $\Sigma|\mathcal{P}_1$.*

Suppose that there is a circuit of finite length in the point-collinearity graph of Σ_1, not contained in the point shadow of a line of Σ. Then $\Sigma_1 = \Sigma$.

Proof. Let $\mathcal{G} = (\mathcal{P}, \mathcal{E})$ and $\mathcal{G}_1 = (\mathcal{P}_1, \mathcal{E}_1)$ be the point-collinearity graphs of Σ and Σ_1 respectively. Since $\mathcal{P}_1$ is a convex subspace of Σ, the graph $\mathcal{G}_1$ is a convex induced subgraph of $\mathcal{G}$. Let w be a circuit of $\mathcal{G}_1$ not contained in the shadow of a line of Σ, and suppose that w has the smallest possible length among all circuits of $\mathcal{G}_1$ not contained in the shadow of a line.

Let X denote the set of vertices of w. By hypothesis $|X|$ is finite, therefore n is finite. The smallest length of a circuit in $\mathcal{G}$, not contained in the shadow of a line, is n. Any circuit in $\mathcal{G}$, not contained in the shadow of a line, must contain two points at distance $n/2$ from each other if n is even or, if n is odd, a point p and two points of some line all of whose points are at distance $(n-1)/2$ from p. In particular, this is true of the circuit w. The graph $\mathcal{G}_1$ is a convex induced subgraph of $\mathcal{G}$, and X is a shortest circuit of $\mathcal{G}_1$, therefore $|X| = n$.

Let $\mathcal{B}$ denote the chamber system of Σ. Then w is the point shadow of an apartment A of $\mathcal{B}$. That is, $X = \mathcal{P}_A$. By Theorem 1.1 the convex subspace closure of $\mathcal{P}_A$ in Σ is $\mathcal{P}$. Since by hypothesis the set $\mathcal{P}_1$ is a convex subspace of Σ, and $\mathcal{P}_A \subseteq \mathcal{P}_1$, we obtain that $\mathcal{P}_1 = \mathcal{P}$. □

Suppose Γ is the augmented J-Grassmann geometry of a building, and suppose Σ is the point-line truncation of Γ. We use Lemma 6.5 to prove Proposition 6.6 below that shows that a subspace of Σ, properly containing the point shadow of a plane of Γ, cannot be a generalized polygon, if the shadow of the plane has finite diameter.

In the proof of Proposition 6.6 we use two results from [6] and [7]. These results were stated for J-Grassmann geometries but remain true for augmented J-Grassmann geometries.

Proposition 6.6. *Let $\mathcal{B}$ be a building of type $M = (m_{ij})$ over I. Let $S_p \subseteq I$. Let Γ be the augmented $(I - S_p)$-Grassmann geometry of $\mathcal{B}$ and let $\Sigma = (\mathcal{P}, \mathcal{L})$ denote the point-line truncation of Γ.*

Let π be a plane of Γ. Let $R = \mathrm{R}_\pi$, and suppose that the point-collinearity graph of the geometry $\Sigma|\mathcal{P}_R$ has finite diameter.

Let $\mathcal{P}_1$ be a subspace of Σ containing $\mathcal{P}_R$. If $\Sigma|\mathcal{P}_1$ is a generalized polygon, then $\mathcal{P}_R = \mathcal{P}_1$.

Proof. By Corollary 6.2 of [6] $\mathcal{P}_R$ is a convex subspace of Σ. Therefore $\mathcal{P}_R$ is a convex subspace of the subspace $\Sigma|\mathcal{P}_1$ of Σ.

Since $\Sigma|\mathcal{P}_R$ is the point shadow of a plane of Γ, by Proposition 3.8 of [7] the subspace $\Sigma|\mathcal{P}_R$ is a generalized n-gon with $n \geq 3$. By hypothesis the point-collinearity graph of $\Sigma|\mathcal{P}_R$ has finite diameter, therefore n is finite. This implies that the point-collinearity graph of $\Sigma|\mathcal{P}_R$ contains a circuit of finite length, not contained in the point shadow of a line of Σ.

We have shown that $\Sigma|\mathcal{P}_R$ and $\Sigma|\mathcal{P}_1$ satisfy the hypothesis of Lemma 6.5. Therefore $\mathcal{P}_R = \mathcal{P}_1$. □

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